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Highest-weight $U_q[\mathrm{sl}(n)]$ modules and invariant integrable n-state models with periodic boundary conditions

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Abstract. Weights are computed for the Bethe vectors of an RSOS-type model with periodic boundary conditions obeying $U_q[\mathrm{sl}(n)]$ ($q = \exp(\mathrm{i}\pi/r)$) invariance. They are shown to be highest-weight vectors. The q-dimensions of the corresponding irreducible representations are obtained.

In recent years, considerable progress has been made on the 'quantum symmetry' of integrable quantum chain models such as the *XXZ* Heisenberg model and its generalizations. In [1], we constructed an $\mathrm{sl}_q(n)$ invariant RSOS-type model with periodic boundary conditions. The model can be regarded as a generalization of the *XXZ*-model with twisted boundary conditions. Therefore, the representational analysis of $U_q[\mathrm{sl}(2)]$, as treated, for example, in [2], has to be extended to cases of higher rank as was first considered in [3].

In the present paper, we prove, for this model, the highest-weight property of the Bethe states, calculate the weights and the q-dimensions of the representations and classify the irreducible representations. For the case of open boundary conditions, see, for example, [4–6].

The model of [1] is defined by the transfer matrix $\tau = \tau^{(n)}$, where

$$\tau^{(k)}(x,\underline{x}^{(k)}) = \mathrm{tr}_q(\mathcal{T}^{(k)}(x,\underline{x}^{(k)}) = \sum_{\alpha} q^{n+1-2\alpha}(\mathcal{T}^{(k)})^{\alpha}_{\alpha}(x,\underline{x}^{(k)}) \qquad k = 1,\ldots,n. \tag{1}$$

The 'doubled' monodromy matrix is given by

$$\mathcal{T}^{(k)}_0(x,\underline{x}^{(k)}) = \tilde{T}^{(k)}_0 \cdot T^{(k)}_0(x,\underline{x}^{(k)}) = (R_{01}\ldots R_{0N_k}) \cdot (R_{N_k0}(x_{N_k}/x^{(k)})\ldots R_{10}(x_1/x^{(k)})). \tag{2}$$

For the nested algebraic Bethe ansatz, in addition to $\mathcal{T}(x) = \mathcal{T}^{(n)}(x)$, the monodromy matrices for all $k \leqslant n$ are needed. The $\mathrm{sl}_q(k)$ R-matrix is given by

$$\begin{aligned} R(x) &= xR - x^{-1}PR^{-1}P \\ R &= \sum_{\alpha\neq\beta} E_{\alpha\alpha}\otimes E_{\beta\beta} + q\sum_{\alpha} E_{\alpha\alpha}\otimes E_{\alpha\alpha} + (q-q^{-1})\sum_{\alpha>\beta} E_{\alpha\beta}\otimes E_{\beta\alpha}. \end{aligned} \tag{3}$$

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The Yang–Baxter equation reads:

$$R_{12}(y/x)\mathcal{T}_1(x)R_{21}\mathcal{T}_2(y) = \mathcal{T}_2(y)R_{12}\mathcal{T}_1(x)R_{21}(y/x). \tag{4}$$

The model is quantum-group invariant, i.e. the transfer matrix commutes with the generators of $U_q[\mathrm{sl}(n)]$. These are obtained from the monodromy matrices $T(x)$ in the limits x to 0 or ∞ (up to normalizations)

$$T = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \alpha E_1 & 1 & \ddots & \vdots \\ & \ddots & \ddots & 0 \\ * & & \alpha E_{n-1} & 1 \end{pmatrix} q^{W} \qquad T_\infty = q^{-W}\begin{pmatrix} 1 & -\alpha F_1 & & * \\ 0 & 1 & \ddots & \\ & \ddots & \ddots & -\alpha F_{n-1} \\ 0 & & 0 & 1 \end{pmatrix} \tag{5}$$

where $\alpha = q - q^{-1}$ and the matrix $W = \mathrm{diag}\{W_1, \dots, W_n\}$ contains the $U_q[\mathrm{gl}(n)]$ Cartan elements. Analogously to equation (5), we introduce

$$\mathcal{T} = \tilde{T} \cdot T \qquad \text{where } \tilde{T} = T_\infty^{-1} \tag{6}$$

as a limit of $\mathcal{T}^{(n)}(x, \underline{x}^{(n)})$ for $x \to 0$, where here, and in the following, operators without argument denote these limits for $x \to 0$.

We write the doubled monodromy matrices as $k \times k$ block-matrices of operators

$$\mathcal{T}^{(k)}(x) = \begin{pmatrix} \mathcal{A}^{(k)}(x) & \mathcal{B}^{(k)}(x) \\ \mathcal{C}^{(k)}(x) & \mathcal{D}^{(k)}(x) \end{pmatrix}. \tag{7}$$

We also introduce the reference states $\Phi^{(k)}$ with $\mathcal{C}^{(k)}(x)\Phi^{(k)} = 0$. The eigenstates of the transfer matrix $\tau(x)$ are the Bethe-ansatz states $\Psi = \Psi^{(n)}$ obtained by the nested procedure

$$\Psi^{(k)} = \mathcal{B}^{(k)}_{\alpha_1}(x_1^{(k-1)}) \dots \mathcal{B}^{(k)}_{\alpha_{N_{k-1}}}(x_{N_{k-1}}^{(k-1)})\Phi^{(k)}\Psi^{(k-1)}_{\underline{\alpha}} \qquad (k = 2, \dots, n) \qquad \Psi^{(1)} = 1. \tag{8}$$

The sets of parameters $x_j^{(k)} = \exp(\mathrm{i}\theta_j^{(k)} - (n-k)\gamma/2)$ $(q = \mathrm{e}^{\mathrm{i}\gamma})$ fulfil the Bethe-ansatz equations: for $j = 1, \dots, N_k$ and $k = 1, \dots, n-1$

$$q^{2+w_{n-k}-w_{n-k+1}} \prod_{l=k\pm1} \prod_{i=1}^{N_l} \frac{\sinh\frac{1}{2}\left(\theta_j^{(k)} - \theta_i^{(l)} - i\gamma\right)}{\sinh\frac{1}{2}\left(\theta_j^{(k)} - \theta_i^{(l)} + i\gamma\right)} \prod_{i=1}^{N_k} \frac{\sinh\frac{1}{2}\left(\theta_j^{(k)} - \theta_i^{(k)} + 2i\gamma\right)}{\sinh\frac{1}{2}\left(\theta_j^{(k)} - \theta_i^{(k)} - 2i\gamma\right)} = -1 \tag{9}$$

where, below, the $w_i = N_{n-i+1} - N_{n-i}$ will turn out to be the weights of state Ψ, i.e. the eigenvalues of the W_i's defined by equation (5).

Theorem. The Bethe-ansatz states are highest-weight states, i.e.

$$E_i\Psi = 0 \qquad (i = 1, \dots, n-1)$$

a statement which was proved for the first time for rank > 1 within the context of the Hubbard model in [7]. The $\mathrm{sl}_q(n)$ symmetric case with open boundary conditions has been treated in [5].

Proof. First, we prove $\mathcal{T}^\alpha_\beta \Psi = 0$ and then $T^\alpha_\beta \Psi = 0$ for $\alpha > \beta$. The Yang–Baxter relation (4) implies

$$\mathcal{T}^\alpha_\beta \mathcal{B}_\gamma(x) = \mathcal{B}_{\gamma'}(x)\mathcal{T}^{\alpha'}_{\beta'} R^{\gamma'\alpha}_{\gamma''\alpha'} R^{\beta'\gamma''}_{\beta\gamma} \qquad \text{(for } \alpha > 1\text{) otherwise see equation (12)).} \tag{10}$$

We apply the technique of the nested algebraic Bethe ansatz and commute $\mathcal{T}$ through all the $\mathcal{B}$'s of Ψ in equation (8)

$$\mathcal{T}^\alpha_\beta \mathcal{B}_{\alpha_1}(x_1)\ldots\mathcal{B}_{\alpha_{N_{n-1}}}(x_{N_{n-1}})\Phi^{(k)}\Psi^{(n-1)}_{\underline{\alpha}} = \mathcal{B}_{\alpha_1}(x_1)\ldots\mathcal{B}_{\alpha_{N_{n-1}}}(x_{N_{n-1}})\Phi^{(k)}(\mathcal{T}^{(n-1)})^\alpha_\beta \Psi^{(n-1)}_{\underline{\alpha}}. \tag{11}$$

Iterating this procedure $\beta - 1$ times, we arrive at $\mathcal{C}^{(k)}_{\alpha'}\Psi^{(k)}$ with $k = n-\beta+1$ and $\alpha' = \alpha-\beta$. At this stage, we use, as usual (see, for example, [8]), the commutation rule

$$\mathcal{C}^{(k)}_{\alpha'}\mathcal{B}^{(k)}_\gamma(x) = q^{-1}R^{\gamma'\alpha'}_{\gamma\alpha''}\mathcal{B}^{(k)}_{\gamma'}(x)\mathcal{C}^{(k)}_{\alpha''} + (1-q^{-2})((\mathcal{D}^{(k)})^{\alpha'}_\gamma \mathcal{A}^{(k)}(x) - (\mathcal{D}^{(k)})^{\alpha'}_{\alpha''}(\mathcal{D}^{(k)})^{\alpha''}_\gamma(x)) \tag{12}$$

to prove that the Bethe-ansatz equations (9) imply

$$\mathcal{C}^{(k)}_{\alpha'}\Psi^{(k)} = 0 \qquad (k = 2,\ldots,n) \qquad \mathcal{T}^\alpha_\beta\Psi = 0 \qquad \text{for } \alpha > \beta. \tag{13}$$

Finally, we show $T^\alpha_\beta \Psi = 0$ for $\alpha > \beta$. We have, from equations (5) and (13), for all $\beta < n$,

$$\mathcal{T}^n_\beta \Psi = \tilde{T}^n_n T^n_\beta \Psi = 0 \tag{14}$$

and because $\tilde{T}^n_n$ is an invertible operator, it follows that $T^n_\beta \Psi = 0$. Now we consider the previous row, where $\beta < n-1$

$$\mathcal{T}^{n-1}_\beta \Psi = (\tilde{T}^{n-1}_{n-1} T^{n-1}_\beta + \tilde{T}^{n-1}_n T^n_\beta)\Psi = 0 \tag{15}$$

and, therefrom, along with the foregoing result, we get $T^{n-1}_\beta \Psi = 0$. By iteration, we find $T^\alpha_\beta \Psi = 0$ for $\alpha > \beta$ and, hence, from equation (5), $E_i\Psi = 0$ for all i. This proves the highest-weight property of the Bethe vectors. □

Next, we compute the weights of the Bethe-ansatz states Ψ. We consider first

$$\mathcal{T}^\alpha_\alpha \Psi = \tilde{T}^\alpha_\alpha T^\alpha_\alpha \Psi + \sum_{\beta>\alpha} \tilde{T}^\alpha_\beta T^\beta_\alpha \Psi. \tag{16}$$

Again shifting $\mathcal{T}^\alpha_\alpha$ to the right, as in equation (11), we get the operator $(\mathcal{T}^{(n-1)})^\alpha_\alpha$. By iteration, we arrive at $\mathcal{A}^{(k)}\Psi^{(k)}$ for $k = n - \alpha + 1$. The Yang–Baxter relation (4) and equations (2) and (3) imply

$$\mathcal{A}^{(k)}\mathcal{B}^{(k)}_\beta(x) = q^{-2}\mathcal{B}^{(k)}_\beta(x)\mathcal{A}^{(k)} \qquad \mathcal{A}^{(k)}\Psi^{(k)} = q^{2N_k}\Psi^{(k)} \tag{17}$$

and, therefore,

$$\mathcal{A}^{(k)}\Psi^{(k)} = q^{2(N_k - N_{k-1})}\Psi^{(k)}. \tag{18}$$

From equations (5) and (6), we have

$$\tilde{T}_i^i = T_i^i = q^{W_i} \qquad \text{and} \qquad \mathcal{T}_i^i = q^{2W_i} \tag{19}$$

and, finally,

$$q^{2W_i}\Psi = q^{2w_i}\Psi \qquad \text{with} \qquad w_i = N_{n-i+1} - N_{n-i}. \tag{20}$$

So, any Bethe-ansatz solution is characterized by a weight vector

$$w = (w_1, \ldots, w_n) = (N_n - N_{n-1}, \ldots, N_2 - N_1, N_1) \tag{21}$$

with the usual highest-weight condition

$$w_1 \geqslant \cdots \geqslant w_n \geqslant 0. \tag{22}$$

Here, $N = N^{(n)}$ is the number of lattice sites and $N^{(k)}$ $(k = n-1, \ldots, 1)$ is the number of roots in the kth Bethe-ansatz level. The highest-weight condition (22) may be shown as usual. The result (21) is consistent with the 'ice rule' fulfilled by the R-matrix (3). This means that each operator $\mathcal{B}_\alpha^{(k)}(x)$ reduces w_k and lifts w_α by one.

The q-dimension of a representation π with representation space V is obtained from the 'Markov trace' (see, for example, [3] and [9])

$$\dim_q \pi = \mathrm{tr}_V(q^{-\sum_{i>j}(W_j - W_i)}). \tag{23}$$

As is well known for the case of q being a root of unity, the generators E_i and F_i become nilpotent

$$(E_i)^r = (F_i)^r = 0 \qquad q = \exp(\mathrm{i}\pi/r) \qquad r = n+2, n+3, \ldots. \tag{24}$$

A highest-weight module is equivalent to the corresponding module of $\mathrm{sl}(n)$, if this relation does not concern it. These representations still remain irreducible and will be called good.

The other representations are called bad and, up to special irreducible cases with vanishing q-dimension, they are reducible but not decomposable.

For the irreducible representations π_w, with highest-weight vector w, equation (23) gives

$$\dim_q \pi_w = \prod_{\alpha\in\Phi_+} \frac{[(w+g,\alpha)]_q}{[(g,\alpha)]_q} = \prod_{i>j} \frac{[w_j - w_i + i - j]_q}{[i-j]_q} \tag{25}$$

where $[x]_q = (q^x - q^{-x})/(q - q^{-1})$ is a q-number, Φ_+ denotes the set of positive roots and g is the Weyl vector $g = \frac{1}{2}\sum_{\alpha\in\Phi_+}\alpha$. The good representations are characterized by positive q-dimensions. From equation (25), it follows that their weight patterns are restricted by their length

$$w_1 - w_n \leqslant r - n. \tag{26}$$

The q-dimensions of bad representations vanish.

It is an interesting question as to how these good representations are characterized in the language of the Bethe ansatz. In [6], it is shown for $\mathrm{sl}_q(2)$ that these good representations are given by all Bethe-ansatz solutions with only positive parity strings (in the language of Takahashi [10]) which are additionally restricted by their length. In a forthcoming paper, we will show how this classification extends to q-symmetries of higher rank.

References

[1] Karowski M and Zapletal A 1994 *Nucl. Phys.* B **419** 567

[2] Karowski M 1990 *Quantum Groups (Springer Lecture Notes in Physics 370)* (Berlin: Springer) p 183
Levy D 1991 *Phys. Rev. Lett.* **67** 1971
Martin P and Saleur H 1994 *Lett. Math. Phys.* **30** 189

[3] Pasquier V and Saleur H 1990 *Nucl. Phys.* B **330** 523

[4] Mezincescu L and Nepomechie R I 1991 *Mod. Phys. Lett.* A **6** 2497
Destri C and de Vega H J 1992 *Nucl. Phys.* B **361** 361
Förster A and Karowski M 1993 *Nucl. Phys.* B **408** 512
de Vega H J and González-Ruiz A 1994 *Nucl. Phys.* B **417** 553

[5] de Vega H J and González-Ruiz A 1994 Highest-weight property for the $SU_q(n)$ invariant spin chains *Preprint* LPTHE-PAR 94/13

[6] Jüttner G and Karowski M 1994 The 'good' Bethe-ansatz solutions of quantum group invariant Heisenberg models *Preprint* SFB 288

[7] Essler F, Korepin V and Schoutens K 1992 *Nucl. Phys.* B **372** 559

[8] de Vega H J 1989 *Int. J. Mod. Phys.* A **4** 2063

[9] Reshetikhin N Yu 1987 Quantized universal enveloping algebras, YBE and invariants of links (Part I) *Preprint* LOMI E-4-87
Reshetikhin N and Turaev V G 1991 *Invent. Math.* **103** 547
Fuchs J 1992 *Affine Lie Algebras and Quantum Groups* (Cambridge: Cambridge University Press)

[10] Takahashi M and Suzuki M 1972 *Prog. Theor. Phys.* **48** 2187